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Nonlinear and Linear α-Weighted Methods for Particle Transport Problems

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A parametrized family of iterative methods for the planar-geometry transport equation is proposed. This family is a generalization of previously proposed nonlinear flux methods. The new methods are derived by integrating the 1D transport equation over $-1 \le \mu \le 0$ and $0 \le \mu \le 1$ with weight $|\mu|^{\alpha}$, $\alpha \ge 0$. Both nonlinear and linear methods are developed. The convergence properties of the proposed methods are studied theoretically by means of a Fourier stability analysis. The optimum value of α that provides the best convergence rate is derived. We also show that the convergence rates of nonlinear and linear methods are almost the same. Numerical results are presented to confirm these theoretical predictions. © 2001 Academic Press

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1. INTRODUCTION

To solve the particle transport equation, iteration methods must be utilized, due to the integro-differential nature of this equation. For many highly scattering problems, the source iteration algorithm converges slowly and requires an inordinate amount of computer time. To accelerate these iterations, nonlinear projective-iteration (NPI) methods [1, 2] have been developed; these are also known as projected discrete ordinates (PDO) methods [3]. Among these methods are the quasi-diffusion (QD) method [4], a method based on the Yvon–Mertens approximation [5], the second-moment method [6], the first-flux (FF) (averaged flux) method [7–9], second-flux (SF) method [10], and nonlinear S_2-like methods [11].

The NPI (PDO) methods have the following features:

1. Most of these methods are nonlinear.
2. Each method is defined by a system of equations consisting of two parts: the "transport" ("high-order") and "low-order" (moments) equations.

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3. The low-order equations are a set of equations for various moments of the distribution function over angles and energy.

4. These methods converge very rapidly.

5. The nonlinear methods contain special functionals that are weakly dependent on the transport solution. This results in fast convergence rates.

6. For stability, there is no need to discretize the transport and low-order equations consistently. These independent discretizations may make it possible to improve the accuracy of the numerical solution.

We note that the QD method has also been successfully used for solving upscattering problems [12].

In general, the discretized equations of an NPI (PDO) method yield two scalar flux solutions. One is generated by the high-order problem, the other by the low-order problem. The accuracies of these two scalar fluxes depend on their spatial discretizations. A “consistent” discretization [13, 14] results in the scalar fluxes being identical, in which case the NPI (PDO) methods are pure acceleration methods (the converged accelerated and unaccelerated solutions are identical on any grid). Independent discretizations [2] generally give rise to scalar fluxes that differ by a truncation error. Thus, an NPI (PDO) method approximated by independent schemes is a rapidly converging iterative method, but not a pure acceleration method (the solution is altered by a small amount). However, if the size of the mesh cells tends to zero and the difference schemes for the transport and low-order equations converge, then these two scalar fluxes tend to each other and to the exact solution of the discrete-ordinates equations.

The low-order equations of the QD and “flux” methods are of different forms. An attractive feature of the flux methods is that the low-order equations are simplified transport equations, which can be discretized using known transport differencing schemes. This fact presents an opportunity to *improve* the accuracy of the numerical solution of a given discretization scheme for the high-order transport equation. This can be done if the discretization scheme for the low-order equations is chosen to be more accurate than the original discretization scheme chosen for the high-order equations. Employing such inconsistent discretization schemes leads to a numerical algorithm in which the converged scalar fluxes are (i) more accurate than the original discrete solution, and (ii) are obtained much more efficiently. Consistent discretizations do not possess the option of improving the accuracy of the solution. This issues was discussed and studied in [1, 2, 11].

In this paper, we present a generalization of the FF and SF methods, which, for single-group problems with isotropic scattering, are obtained by integrating the transport equation over the half-ranges $-1 \leq \mu < 0$ and $0 < \mu \leq 1$, with weights 1 and μ, respectively (μ is the direction cosine). These methods are nonlinear. (In a variant of the flux methods, useful for electron transport problems with highly anisotropic scattering, the transport equation is integrated over a number of subintervals of $-1 \leq \mu \leq 1$. [15]).

Here, we develop an expanded nonlinear family of “α-weighted” iteration methods that use weights $|\mu|^\alpha$, where α is any nonnegative constant. If $\alpha = 0$, the FF method results, and if $\alpha = 1$ the SF method results. For other values of α, the iteration methods are new. We derive both nonlinear and linear versions of these methods, which have a similar mathematical structure to the FF and SF methods (the low-order equations are a simplified transport equation). Since the resulting iteration schemes qualify as NPI (PDO) methods, they possess the features described above. Thus, they should have the capacity to produce iterative solutions much more efficiently than by the unaccelerated Source Iteration procedure, and they should have the capacity to produce more accurate solutions under the right circumstances.

In numerical simulations described in this paper, we use the Step Characteristic (SC) spatial differencing scheme [16] for the high-order transport equation, and the more accurate Linear Discontinuous (LD) scheme [17–19] for the low-order equations. The (high-order) SC method is used in several common reactor physics codes. This discretization method has many advantages: it is second-order accurate, and it produces positive and monotonic (nonoscillatory) solutions. However, it is not accurate for diffusive problems with spatial cells that are not optically thin [20]. The (low-order) LD method is more accurate and expensive than SC, but it performs well in diffusive problems with spatial cells that are not optically thin [19]. In this paper, we demonstrate that the scalar fluxes obtained by the resulting α-weighted methods are obtained efficiently, and are more accurate than the pure SC solutions for diffusive problems in which the spatial cells are not optically thin.

This paper is focused on a description of the proposed methods and a consideration of their basic features, namely, stability properties. As a result of theoretical studies, we have determined the nonlinear and linear methods with the optimal values of α (i.e., with the fastest convergence rates). To perform this study, we used a Fourier stability analysis that has been successfully employed for the analysis of various other linear iteration schemes for transport problems [21, 22]. To study nonlinear methods, we analyze the nonlinear methods after they are linearized around a certain simple solution [23]. We confirm our theoretical predictions by numerical results.

The remainder of this paper is organized as follows. In Section 2 we formulate the proposed methods. In Section 3 we study the methods in differential form by means of the Fourier analysis and show that the methods with prescribed values of α have particular properties. In Section 4 we present difference schemes for the considered methods. In Section 5 we perform a theoretical and numerical investigation of the discretized methods applying the Fourier stability analysis. We conclude with a discussion in Section 6.

2. FORMULATION OF α-WEIGHTED METHODS

2.1. Nonlinear Methods

Let us consider the slab geometry transport problem

$$\mu\frac{\partial}{\partial x}\psi(x,\mu)+\sigma_t(x)\psi(x,\mu)=\frac{1}{2}\left(\sigma_s(x)\int_{-1}^{1}\psi(x,\mu')\,d\mu'+Q(x)\right),\quad -1\le\mu\le 1, \tag{1}$$

$$\psi(0,\mu)=\psi(0,-\mu),\quad 0\le\mu<1, \tag{2}$$

$$\psi(L,\mu)=F(\mu),\quad -1\le\mu<0. \tag{3}$$

Here $\psi(x,\mu)$ is the angular flux, σ_t and σ_s are the total and scattering cross sections, respectively, and Q is a source.

To solve this problem by an α-weighted method, we let $\alpha \geq 0$ be a user-defined constant, and we define the quantities:

$$\phi^-(x) = \int_{-1}^{0} \psi(x,\mu)\,d\mu, \tag{4}$$

$$\phi^+(x) = \int_{0}^{1} \psi(x,\mu)\,d\mu, \tag{5}$$

$$\phi(x) = \int_{-1}^{1} \psi(x,\mu)\,d\mu, \tag{6}$$

$$A^-(x) = \frac{(\alpha+1)\int_{-1}^{0} |\mu|^{\alpha+1}\psi(x,\mu)\,d\mu}{\int_{-1}^{0} \psi(x,\mu)\,d\mu}, \tag{7}$$

$$A^+(x) = \frac{(\alpha+1)\int_{0}^{1} \mu^{\alpha+1}\psi(x,\mu)\,d\mu}{\int_{0}^{1} \psi(x,\mu)\,d\mu}, \tag{8}$$

$$B^-(x) = \frac{(\alpha+1)\int_{-1}^{0} |\mu|^{\alpha}\psi(x,\mu)\,d\mu}{\int_{-1}^{0} \psi(x,\mu)\,d\mu}, \tag{9}$$

$$B^+(x) = \frac{(\alpha+1)\int_{0}^{1} \mu^{\alpha}\psi(x,\mu)\,d\mu}{\int_{0}^{1} \psi(x,\mu)\,d\mu}. \tag{10}$$

Operating on Eq. (1) by $(\alpha+1)\int_0^1 \mu^\alpha(\cdot)\,d\mu$ and $(\alpha+1)\int_{-1}^0 |\mu|^\alpha(\cdot)\,d\mu$, and using the functionals defined in Eqs. (7)–(10), we obtain

$$-\frac{d}{dx}A^-\phi^- + (\sigma_t B^- - 0.5\sigma_s)\phi^- = 0.5(\sigma_s\phi^+ + Q), \tag{11}$$

$$\frac{d}{dx}A^+\phi^+ + (\sigma_t B^+ - 0.5\sigma_s)\phi^+ = 0.5(\sigma_s\phi^- + Q). \tag{12}$$

Likewise, operating on Eq. (2) by $\int_0^1(\cdot)\,d\mu$ and on Eq. (3) by $\int_{-1}^0(\cdot)\,d\mu$, we obtain

$$\phi^+(0) = \phi^-(0), \tag{13}$$

$$\phi^-(L) = \phi^*, \tag{14}$$

where

$$\phi^* = \int_{-1}^{0} F(\mu)\,d\mu.$$

The α-weighted nonlinear (α-WN) method is defined by Eqs. (1)–(14). This set of equations is solved by the iteration scheme

$$\mu\frac{\partial}{\partial x}\psi^{(k+1/2)} + \sigma_t\psi^{(k+1/2)} = 0.5\big(\sigma_s\phi^{(k)} + Q\big), \tag{15}$$

$$\psi^{(k+1/2)}(0,\mu) = \psi^{(k+1/2)}(0,-\mu), \tag{16}$$

$$\psi^{(k+1/2)}(L,\mu) = F(\mu), \quad -1 \le \mu < 0, \tag{17}$$

$$A^{-(k+1/2)} = \frac{(\alpha+1)\int_{-1}^{0} |\mu|^{\alpha+1}\psi^{(k+1/2)}d\mu}{\int_{-1}^{0}\psi^{(k+1/2)}\,d\mu}, \tag{18}$$

$$A^{+(k+1/2)} = \frac{(\alpha+1)\int_{0}^{1} \mu^{\alpha+1}\psi^{(k+1/2)}d\mu}{\int_{0}^{1}\psi^{(k+1/2)}\,d\mu}, \tag{19}$$

$$B^{-(k+1/2)} = \frac{(\alpha+1)\int_{-1}^{0} |\mu|^{\alpha}\psi^{(k+1/2)}d\mu}{\int_{-1}^{0}\psi^{(k+1/2)}\,d\mu}, \tag{20}$$

$$B^{+(k+1/2)} = \frac{(\alpha+1)\int_{0}^{1} \mu^{\alpha}\psi^{(k+1/2)}d\mu}{\int_{0}^{1}\psi^{(k+1/2)}\,d\mu}, \tag{21}$$

$$-\frac{d}{dx}A^{-(k+1/2)}\phi^{-(k+1)} + \left(\sigma_t B^{-(k+1/2)} - 0.5\sigma_s\right)\phi^{-(k+1)} = 0.5\left(\sigma_s\phi^{+(k+1)} + Q\right) \tag{22}$$

$$\frac{d}{dx}A^{+(k+1/2)}\phi^{+(k+1)} + \left(\sigma_t B^{+(k+1/2)} - 0.5\sigma_s\right)\phi^{+(k+1)} = 0.5\left(\sigma_s\phi^{-(k+1)} + Q\right), \tag{23}$$

$$\phi^{(k+1)} = \phi^{+(k+1)} + \phi^{-(k+1)}, \tag{24}$$

$$\phi^{+(k+1)}(0) = \phi^{-(k+1)}(0), \tag{25}$$

$$\phi^{-(k+1)}(L) = \phi^{*}, \tag{26}$$

where k is the iteration index. For the initial iteration, $\phi^{-(0)}$ and $\phi^{+(0)}$ are determined by solving the low-order problem (22)–(26) with $A^{\pm} = \frac{\alpha+1}{\alpha+2}$ and $B^{\pm} = 1$. Thus, the above nonlinear iteration scheme consists of three parts:

1. A conventional transport sweep to calculate $\psi^{(k+1/2)}(x,\mu)$ (Eqs. (15)–(17)).
2. The calculation of the functionals $A^{\pm(k+1/2)}(x)$ and $B^{\pm(k+1/2)}(x)$ from $\psi^{(k+1/2)}(x,\mu)$ (Eqs. (18)–(21)).
3. Solving a low-order transport problem (Eqs. (22)–(26)) for $\phi^{\pm(k+1)}(x)$.

For $\alpha = 0$, the above method reduces to the FF method [8, 9], and for $\alpha = 1$ it reduces to the SF method [10]. Thus, the FF and SF methods are special cases of the α-WN methods.

2.2. Linear Methods

Here we shall derive linear versions of the nonlinear methods developed above. We again consider problem (1)–(3), and we operate by $(\alpha+1)\int_0^1 \mu^{\alpha}(\cdot)\,d\mu$ and $(\alpha+1)\int_{-1}^{0}|\mu|^{\alpha}(\cdot)\,d\mu$ to obtain

$$-(\alpha+1)\frac{d}{dx}\left(\int_{-1}^{0}|\mu|^{\alpha+1}\psi(x,\mu)\,d\mu\right) + (\alpha+1)\sigma_t(x)\int_{-1}^{0}|\mu|^{\alpha}\psi(x,\mu)\,d\mu$$
$$= 0.5(\sigma_s(x)(\phi^-(x)+\phi^+(x)) + Q(x)), \tag{27}$$

$$(\alpha+1)\frac{d}{dx}\left(\int_{0}^{1}\mu^{\alpha+1}\psi(x,\mu)\,d\mu\right) + (\alpha+1)\sigma_t(x)\int_{0}^{1}\mu^{\alpha}\psi(x,\mu)\,d\mu$$
$$= 0.5(\sigma_s(x)(\phi^-(x)+\phi^+(x)) + Q(x)). \tag{28}$$

Then, we define

$$P^-(x) = \int_{-1}^{0} [(\alpha + 1)|\mu|^\alpha - 1]\psi(x, \mu)\, d\mu, \tag{29}$$

$$P^+(x) = \int_{0}^{1} [(\alpha + 1)\mu^\alpha - 1]\psi(x, \mu)\, d\mu, \tag{30}$$

$$R^-(x) = \int_{-1}^{0} [(\alpha + 2)|\mu|^{\alpha+1} - 1]\psi(x, \mu)\, d\mu, \tag{31}$$

$$R^+(x) = \int_{0}^{1} [(\alpha + 2)\mu^{\alpha+1} - 1]\psi(x, \mu)\, d\mu, \tag{32}$$

$$\tilde{\sigma} = \sigma_t - 0.5\sigma_s, \tag{33}$$

$$\gamma = \frac{\alpha + 1}{\alpha + 2}. \tag{34}$$

Rearranging, we obtain

$$-\gamma \frac{d\phi^-}{dx} + \tilde{\sigma}\phi^- = \frac{1}{2}(\sigma_s \phi^+ + Q) - \sigma_t P^- + \gamma \frac{dR^-}{dx}, \tag{35}$$

$$\gamma \frac{d\phi^+}{dx} + \tilde{\sigma}\phi^+ = \frac{1}{2}(\sigma_s \phi^- + Q) - \sigma_t P^+ - \gamma \frac{dR^+}{dx}. \tag{36}$$

Note that we have added terms to both sides of the equations, and that the last two terms on the right sides of Eqs. (35) and (36) vanish if $\psi(x, \mu)$ is constant for $\mu < 0$ and constant for $\mu > 0$. The boundary conditions for the low-order equations are the same as in the α-WN methods; i.e.,

$$\phi^+(0) = \phi^-(0), \tag{37}$$

$$\phi^-(L) = \phi^*. \tag{38}$$

Thus, the α-weighted linear (α-WL) methods are defined by Eqs. (1)–(3) and (29)–(38). The iteration scheme for the α-WL methods is similar to that for the α-WN methods. It consists of three parts:

1. A transport sweep to calculate $\psi^{(k+1/2)}(x, \mu)$ (Eqs. (1)–(3)).
2. The calculation of the functionals $P^{\pm^{(k+1/2)}}(x)$ and $R^{\pm^{(k+1/2)}}(x)$ from $\psi^{(k+1/2)}(x, \mu)$ (Eqs. (29)–(32)).
3. Solving a low-order transport problem (Eqs. (35)–(38)) for $\phi^{\pm^{(k+1)}}(x)$.

For the initial iteration, $\phi^{-^{(0)}}$ and $\phi^{+^{(0)}}$ are determined by solving the low-order problem (35)–(38) with $P^\pm = 0$ and $R^\pm = 0$.

3. ANALYSIS OF THE METHODS IN DIFFERENTIAL FORM

3.1. Linearized α-Weighted Nonlinear Methods

In order to study the convergence properties of the proposed nonlinear methods, we shall linearize them [23]. We consider an infinite medium, constant cross section, and flat source problem. The exact solution of this problem is simply

$$\psi(x,\mu) = \frac{Q}{2\sigma_a}, \quad \text{where } \sigma_a = \sigma_t - \sigma_s. \tag{39}$$

We assume that the iterations begin with an initial estimate that is close to this exact solution, and we assume that all succeeding iterations remain close to this solution. Consequently, we set

$$\psi^{(k+1/2)}(x,\mu) = \frac{Q}{\sigma_a}\left(\frac{1}{2} + \varepsilon\xi^{(k+1/2)}(x,\mu)\right), \tag{40}$$

$$\phi^{\pm^{(k)}}(x) = \frac{Q}{\sigma_a}\left(\frac{1}{2} + \varepsilon\zeta^{\pm^{(k)}}(x)\right), \tag{41}$$

$$\psi^{(k)}(x) = \frac{Q}{\sigma_a}\left(1 + \varepsilon\zeta^{(k)}(x)\right), \tag{42}$$

where $\varepsilon \ll 1$. The rate at which $\xi^{(k+1/2)}(x,\mu)$, $\zeta^{\pm^{(k)}}(x)$ and $\zeta^{(k)}(x)$ tend to zero gives the convergence rate of the method. If these quantities grow rather than decay, then the method is unstable.

To analyze the α-WN methods, we introduce the ansatz, (40)–(42), for the angular and scalar fluxes into Eqs. (15) and (22)–(24). Then we expand in powers of ε and drop the $O(\varepsilon^2)$ terms. The O(1) terms are automatically satisfied and the $O(\varepsilon)$ equations yield

$$\mu\frac{\partial}{\partial x}\xi^{(k+1/2)} + \sigma_t\xi^{(k+1/2)} = 0.5\sigma_s\zeta^{(k)}, \tag{43}$$

$$-\gamma\frac{d}{dx}\zeta^{-^{(k+1)}} + \tilde{\sigma}\zeta^{-^{(k+1)}} = 0.5\sigma_s\zeta^{+^{(k+1)}} - \sigma_t\eta_0^{-(k+1/2)} + \gamma\frac{d}{dx}\eta_1^{-(k+1/2)}, \tag{44}$$

$$\gamma\frac{d}{dx}\zeta^{+^{(k+1)}} + \tilde{\sigma}\zeta^{+^{(k+1)}} = 0.5\sigma_s\zeta^{-^{(k+1)}} - \sigma_t\eta_0^{+(k+1/2)} - \gamma\frac{d}{dx}\eta_1^{+(k+1/2)}, \tag{45}$$

$$\zeta^{(k+1)} = \zeta^{-^{(k+1)}} + \zeta^{+^{(k+1)}}, \tag{46}$$

$$\eta_n^{-(k+1/2)}(x) = \int_{-1}^{0}[(\alpha+1+n)|\mu|^{\alpha+n} - 1]\xi^{(k+1/2)}(x,\mu)\,d\mu, \quad n = 0, 1. \tag{47}$$

$$\eta_n^{+(k+1/2)}(x) = \int_{0}^{1}[(\alpha+1+n)\mu^{\alpha+n} - 1]\xi^{(k+1/2)}(x,\mu)\,d\mu, \quad n = 0, 1. \tag{48}$$

The system of the linearized equations consists of Eqs. (43)–(48). Using the same approach, we obtain equations for the variations ξ, $\zeta^{\pm}$, ζ for the α-WL methods. The result is exactly Eqs. (43)–(48). Thus, the linear and linearized methods are identical in form and have identical convergence properties.

TABLE I
Spectral Radii versus α for c = 1

Method	α	ρ
First-flux	0	0.3333
Optimal	0.128	0.1865
DSA-like	0.366	0.2247
Second-flux	1	0.3105

3.2. Fourier Analysis

We now study the convergence of the linear and linearized nonlinear methods by means of a Fourier analysis. Let us consider a model infinite-medium problem with constant cross sections. We introduce the Fourier ansatz,

$$\xi^{(k+1/2)}(x,\mu) = a(\mu)\omega^k e^{i\lambda\sigma_t x}, \tag{49}$$

$$\zeta^{\pm^{(k)}}(x) = \nu^{\pm}\omega^k e^{i\lambda\sigma_t x}, \tag{50}$$

$$\zeta^{(k)}(x) = \omega^k e^{i\lambda\sigma_t x} \tag{51}$$

into Eqs. (43)–(48) and obtain a system of equations for $a(\mu)$, $\nu^{\pm}$, and ω. Solving this system for the case $\sigma_t = 1$ and $\sigma_s = c$, we obtain

$$\omega = \omega(\lambda) = c\left(\frac{1}{\lambda^2\gamma^2 + 1 - c}\right)\left[(1+\lambda^2\gamma^2)\frac{\tan^{-1}\lambda}{\lambda} - 1\right]. \tag{52}$$

It is clear that $|\omega|$ is maximized for $c = 1$. We note that if $\gamma = \frac{1}{\sqrt{3}}$, i.e., $\alpha = \alpha_{DSA}$, where

$$\alpha_{DSA} = \frac{2-\sqrt{3}}{\sqrt{3}-1} \approx 0.366, \tag{53}$$

then Eq. (52) reduces to exactly the formula that applies to DSA and the linearized quasi-diffusion methods. Hereafter, we refer to the α-WN and α-WL methods with $\alpha = \alpha_{DSA}$ as DSA-like methods.

The spectral radius is defined by

$$\rho = \sup_{\lambda}|\omega(\lambda)|. \tag{54}$$

The values of ρ are tabulated for $c = 1$ and four values of α in Table I. The spectral radius is minimized for $\alpha \approx 0.128$.

These theoretical results show that the performance of various methods depends on the choice of α. However, in practical simulations, the characteristics of each method also depend on the choice of a spatial discretization and an angular quadrature set. In the following, we consider the proposed methods in discretized form.

4. DISCRETIZATION OF α-WEIGHTED METHODS

There are two general approaches for approximation of the equations of the α-weighted methods: consistent and independent schemes [2, 13, 14]. The consistent schemes guarantee the stability of iteration methods for the transport equation [25]. The study of the QD and flux methods showed that it is not necessary to discretize low-order equations of these methods consistently with the transport equation [2, 4, 10, 11, 24], whereas the DSA method requires consistent discretization for stability of iterations [14, 21]. It was theoretically shown that the QD method, approximated by means of some independent difference schemes, converges on a class of model problems, provided that the initial guess is sufficiently close to the converged solution [23]. However, there is no general theorem in this regard for this class of methods. Hence, the stability analysis of the α-weighted methods discretized by independent schemes is of particular interest in the study of their convergence properties.

We now discretize spatially the α-weighted methods using independent differencing schemes for the transport and low-order equations. For the low-order equations, we choose a linear discontinuous (LD) method [17–19]. We introduce a spatial mesh, selected so that $x_{j+1/2}$ $(1 \le j \le J)$ correspond to the mesh edges, where $x_{1/2} = 0$ and $x_{J+1/2} = L$, $x_j = 0.5(x_{j+1/2} + x_{j-1/2})$ are cell centers. The mesh widths are given by $h_j = x_{j+1/2} - x_{j-1/2}$. Integer $\pm\frac{1}{2}$ subscripts refer to cell-edge quantities, and integer subscripts refer to cell-average quantities. The low-order equations of the α-WN method, Eqs. (11)–(14), become

$$-\left(A^-_{j+1/2}\phi^-_{j+1/2} - A^-_{j-1/2}\phi^-_{j-1/2}\right) + (\sigma_{t,j}B^-_j - 0.5\sigma_{s,j})\phi^-_j h_j = 0.5h_j(\sigma_{s,j}\phi^+_j + Q_j), \tag{55}$$

$$A^+_{j+1/2}\phi^+_{j+1/2} - A^+_{j-1/2}\phi^+_{j-1/2} + (\sigma_{t,j}B^+_j - 0.5\sigma_{s,j})\phi^+_j h_j = 0.5h_j(\sigma_{s,j}\phi^-_j + Q_j), \tag{56}$$

$$\begin{aligned} -\theta_j\left(A^-_{j+1/2}\phi^-_{j+1/2} + A^-_{j-1/2}\phi^-_{j-1/2} - 2A^-_j\phi^-_j\right) + (\sigma_{t,j}\{\widehat{B^-\phi^-}\}_j - 0.5\sigma_{s,j}\hat{\phi}^-_j)h_j \\ = 0.5h_j(\sigma_{s,j}\hat{\phi}^+_j + \hat{Q}_j), \end{aligned} \tag{57}$$

$$\begin{aligned} \theta_j\left(A^+_{j+1/2}\phi^+_{j+1/2} + A^+_{j-1/2}\phi^+_{j-1/2} - 2A^+_j\phi^+_j\right) + (\sigma_{t,j}\{\widehat{B^+\phi^+}\}_j - 0.5\sigma_{s,j}\hat{\phi}^+_j)h_j \\ = 0.5h_j(\sigma_{s,j}\hat{\phi}^-_j + \hat{Q}_j), \end{aligned} \tag{58}$$

$$\phi^+_{1/2} = \phi^-_{1/2}, \tag{59}$$

$$\phi^-_{J+1/2} = \phi^*, \tag{60}$$

where

$$\hat{\phi}^-_j = \phi^-_j - \phi^-_{j-1/2}, \tag{61}$$

$$\hat{\phi}^+_j = \phi^+_{j+1/2} - \phi^+_j, \tag{62}$$

$$\{\widehat{B^-\phi^-}\}_j = B^-_j\phi^-_j - B^-_{j-1/2}\phi^-_{j-1/2}, \tag{63}$$

$$\{\widehat{B^+\phi^+}\}_j = B^+_{j+1/2}\phi^+_{j+1/2} - B^+_j\phi^+_j, \tag{64}$$

$$A^\pm_j = 0.5\left(A^\pm_{j+1/2} + A^\pm_{j-1/2}\right), \tag{65}$$

$$B^\pm_j = 0.5\left(B^\pm_{j+1/2} + B^\pm_{j-1/2}\right), \tag{66}$$

$$\hat{Q}_j = \frac{6}{h_j^2}\int_{x_{j-1/2}}^{x_{j+1/2}} (x - x_j)Q(x)\,dx. \tag{67}$$

Equations (55)–(58) are obtained by integrating Eqs. (11) and (12) over the jth cell with weights 1 and $x - x_j$. Here θ_j is a "lumping" parameter; $\theta_j = 3$ corresponds to the standard LD method; $\theta_j = 1$ corresponds to a lumped LD method [19]. The lumped scheme is often used for optically thick cells.

The discretized low-order equations of the α-WL methods are defined by

$$-\gamma\left(\phi^-_{j+1/2} - \phi^-_{j-1/2}\right) + \tilde{\sigma}_j h_j \phi^-_j = 0.5h_j(\sigma_{s,j}\phi^+_j + Q_j) - \sigma_{t,j} h_j P^-_j + \gamma\left(R^-_{j+1/2} - R^-_{j-1/2}\right), \tag{68}$$

$$\gamma\left(\phi^+_{j+1/2} - \phi^+_{j-1/2}\right) + \tilde{\sigma}_j h_j \phi^+_j = 0.5h_j(\sigma_{s,j}\phi^-_j + Q_j) - \sigma_{t,j} h_j P^+_j - \gamma\left(R^+_{j+1/2} - R^+_{j-1/2}\right), \tag{69}$$

$$-\theta_j\gamma\left(\phi^-_{j+1/2} + \phi^-_{j-1/2} - 2\phi^-_j\right) + \tilde{\sigma}_j h_j \hat{\phi}^-_j = 0.5h_j(\sigma_{s,j}\hat{\phi}^+_j + \hat{Q}_j) - \sigma_{t,j} h_j \hat{P}^-_j + \theta_j\gamma\left(R^-_{j+1/2} - R^-_{j-1/2} - 2R^-_j\right), \tag{70}$$

$$\theta_j\gamma\left(\phi^-_{j+1/2} + \phi^-_{j-1/2} - 2\phi^+_j\right) + \tilde{\sigma}_j h_j \hat{\phi}^+_j = 0.5h_j(\sigma_{s,j}\hat{\phi}^-_j + \hat{Q}_j) - \sigma_{t,j} h_j \hat{P}^+_j - \theta_j\gamma\left(R^+_{j+1/2} + R^+_{j-1/2} - 2R^+_j\right), \tag{71}$$

$$\phi^+_{1/2} = \phi^-_{1/2}, \tag{72}$$

$$\phi^-_{J+1/2} = \phi^*, \tag{73}$$

where

$$\hat{\phi}^-_j = \phi^-_j - \phi^-_{j-1/2}, \tag{74}$$

$$\hat{\phi}^+_j = \phi^+_{j+1/2} - \phi^+_j, \tag{75}$$

$$\hat{P}^-_j = P^-_j - P^-_{j-1/2}, \tag{76}$$

$$\hat{P}^+_j = P^+_{j+1/2} - P^+_j, \tag{77}$$

$$P^\pm_j = 0.5\left(P^\pm_{j+1/2} + P^\pm_{j-1/2}\right), \tag{78}$$

$$R^\pm_j = 0.5\left(R^\pm_{j+1/2} + R^\pm_{j-1/2}\right). \tag{79}$$

The approximation of the right-hand sides was performed following the ideas of the LD method, so that the discretized linear and linearized methods are of the same form. Note that the terms with factor θ_j in the right-hand sides of Eqs. (70) and (71) vanish because of Eqs. (79).

To solve the high-order transport equation in the framework of the α-WN methods, a positive monotone differencing scheme is necessary to maintain the stability. For this purpose, we choose the step characteristic (SC) method, which can be written [16]

$$\mu_m\left(\psi_{m,j+1/2} - \psi_{m,j+1/2}\right) + \sigma_{t,j} h_j\left(T_{m,j}\psi_{m,j-1/2} + (1 - T_{m,j})\psi_{m,j+1/2}\right) = 0.5h_j(\sigma_{s,j}\phi_j + Q_j), \tag{80}$$

$$T_{m,j} = \frac{1}{\tau_{m,j}} - \frac{1}{e^{\tau_{m,j}} - 1}, \quad \text{where } \tau_{m,j} = \frac{\sigma_{t,j} h_j}{\mu_m}. \tag{81}$$

The subscript m denotes the discrete ordinates number, where μ_m are the direction cosines $(m = 1, \ldots, M)$. For boundary conditions, we prescribe the incident angular flux on the right boundary,

$$\psi_{m,J+1/2} = F_m, \quad \mu_m < 0, \tag{82}$$

and for a reflective left boundary, we prescribe

$$\psi_{m,1/2} = \psi_{n,1/2}, \quad \text{for } \mu_m = -\mu_n. \tag{83}$$

The same scheme is used for the α-WL methods.

To calculate the functionals $A^{\pm}$, $B^{\pm}$, $P^{\pm}$, and $R^{\pm}$, one should use a quadrature set with angular weights w_m that satisfy the conditions:

$$\sum_{m\in M^-} w_m = \sum_{m\in M^+} w_m = 1, \tag{84}$$

where

$$M^- = \{m : \mu_m \leq 0\}, \quad M^+ = \{m : \mu_m \geq 0\}. \tag{85}$$

The approximation of these functionals will be discussed later.

Note that in optically thick diffusion regions, the SC method generates an isotropic angular flux, and this results in correct values of functionals. For the low-order equations of the α-weighted methods, the LD discretization gives rise to accurate approximation of the diffusion equation in optically thick regions in case of $\alpha = 0.366$ (i.e., DSA-like methods).

5. ANALYSIS OF THE DISCRETIZED METHODS

5.1. Linearization

To study the convergence properties of the discretized nonlinear methods, we use the same approach as above. We linearize the discretized equations, considering again an infinite medium, constant cross section, and a flat source problem with a uniform mesh. The exact solution of this problem is

$$\psi_{m,j+1/2} = \frac{Q}{2\sigma_a}, \quad \text{where } \sigma_a = \sigma_t - \sigma_s. \tag{86}$$

We assume that the iterations begin with an initial estimate that is close to the exact solution (86). Thus, we set

$$\psi^{(k+1/2)}_{m,j+1/2} = \frac{Q}{\sigma_a}\left(\frac{1}{2} + \varepsilon\xi^{(k+1/2)}_{m,j+1/2}\right), \tag{87}$$

$$\phi^{\pm^{(k)}}_{j+1/2} = \frac{Q}{\sigma_a}\left(\frac{1}{2} + \varepsilon\zeta^{\pm^{(k)}}_{j+1/2}\right), \tag{88}$$

$$\phi^{\pm^{(k)}}_{j} = \frac{Q}{\sigma_a}\left(\frac{1}{2} + \varepsilon\zeta^{\pm^{(k)}}_{j}\right), \tag{89}$$

$$\phi^{(k)}_{j} = \frac{Q}{\sigma_a}\left(1 + \varepsilon\zeta^{(k)}_{j}\right), \tag{90}$$

where $\varepsilon \ll 1$.

To analyze the α-WN methods, we introduce the ansatz (87)–(90) for the angular and scalar fluxes into Eqs. (55)–(67), (80), and (81), taking into account the iteration scheme (15)–(26). Then we expand the resulting equations in powers of ε, and drop the $O(\varepsilon^2)$ terms. In the transport differencing scheme (80) and (81), the O(1) terms are automatically satisfied and the $O(\varepsilon)$ equations yield

$$\mu_m\left(\xi_{m,j+1/2}^{(k+1/2)} - \xi_{m,j-1/2}^{(k+1/2)}\right) + \sigma_t h\left(T_m \xi_{m,j-1/2}^{(k+1/2)} + (1-T_m)\xi_{m,j+1/2}^{(k+1/2)}\right) = 0.5\sigma_s h \zeta_j^{(k)}. \quad (91)$$

In the low-order equations, the O(1) terms cancel out, provided that the quadrature set satisfies the following conditions:

$$\sum_{m\in M^-} |\mu_m|^\alpha w_m = \sum_{m\in M^+} \mu_m^\alpha w_m = \frac{1}{\alpha+1}, \quad \text{for any } \alpha. \quad (92)$$

To circumvent this strong restriction on the quadrature set, we replace the factor $\alpha+1$ in the functionals $A^\pm$ and $B^\pm$ by the factor

$$\alpha_0 = \left(\sum_{m\in M^+} \mu_m^\alpha w_m\right)^{-1}, \quad (93)$$

that is, by an approximation of $\alpha+1$ by means of the utilizing quadrature set. We also consider quadrature sets that meet the symmetry condition

$$\sum_{m\in M^-} |\mu_m|^\alpha w_m = \sum_{m\in M^+} \mu_m^\alpha w_m. \quad (94)$$

Thus, we have

$$A_{j+1/2}^\pm = \frac{\alpha_0 \sum_{m\in M^\pm} |\mu_m|^{\alpha+1} \psi_{m,j+1/2} w_m}{\sum_{m\in M^\pm} \psi_{m,j+1/2} w_m}, \quad (95)$$

$$B_{j+1/2}^\pm = \frac{\alpha_0 \sum_{m\in M^\pm} |\mu_m|^{\alpha} \psi_{m,j+1/2} w_m}{\sum_{m\in M^\pm} \psi_{m,j+1/2} w_m}. \quad (96)$$

As a result, the O(1) terms cancel out and we obtain the $O(\varepsilon)$ equations

$$-\beta_1\left(\zeta_{j+1/2}^{-(k+1)} - \zeta_{j-1/2}^{-(k+1)}\right) + \bar{\sigma} h \zeta_j^{-(k+1)} - 0.5\sigma_s h \zeta_j^{+(k+1)}$$
$$= 0.5\left(\eta_{1,j+1/2}^{-(k+1/2)} - \eta_{1,j-1/2}^{-(k+1/2)} - 0.5\sigma_t h\left(\eta_{0,j-1/2}^{-(k+1/2)} + \eta_{0,j+1/2}^{-(k+1/2)}\right)\right), \quad (97)$$

$$\beta_1\left(\zeta_{j+1/2}^{+(k+1)} - \zeta_{j-1/2}^{+(k+1)}\right) + \tilde{\sigma} h \zeta_j^{+(k+1)} - 0.5\sigma_s h \zeta_j^{-(k+1)}$$
$$= 0.5\left(\eta_{1,j-1/2}^{+(k+1/2)} - \eta_{1,j+1/2}^{+(k+1/2)} - 0.5\sigma_t h\left(\eta_{0,j-1/2}^{+(k+1/2)} + \eta_{0,j+1/2}^{+(k+1/2)}\right)\right), \quad (98)$$

$$-\theta\beta_1\left(\zeta_{j+1/2}^{-(k+1)} + \zeta_{j-1/2}^{-(k+1)} - 2\zeta_j^{-(k+1)}\right) + \tilde{\sigma} h\left(\zeta_j^{-(k+1)} - \zeta_{j-1/2}^{-(k+1)}\right) - 0.5\sigma_s h\left(\zeta_{j+1/2}^{+(k+1/2)} - \zeta_j^{+(k+1)}\right)$$
$$= 0.25\sigma_t h\left(\eta_{0,j-1/2}^{-(k+1/2)} - \eta_{0,j+1/2}^{-(k+1/2)}\right), \quad (99)$$

$$\theta\beta_1\left(\zeta_{j+1/2}^{+(k+1)} + \zeta_{j-1/2}^{+(k+1)} - 2\zeta_j^{+(k+1)}\right) + \tilde{\sigma} h\left(\zeta_{j+1/2}^{+(k+1)} - \zeta_j^{+(k+1)}\right) - 0.5\sigma_s h\left(\zeta_j^{-(k+1)} - \zeta_{j-1/2}^{-(k+1)}\right)$$
$$= 0.25\sigma_t h\left(\eta_{0,j-1/2}^{+(k+1/2)} - \eta_{0,j+1/2}^{+(k+1/2)}\right), \quad (100)$$

$$\zeta_j^{(k+1)} = \zeta_j^{-(k+1)} + \zeta_j^{+(k+1)}, \quad (101)$$

where, for $n = 0$ and 1,

$$\eta^{\pm(k+1/2)}_{n,j+1/2} = 2 \sum_{m \in M^{\pm}} (\alpha_0 |\mu_m|^{\alpha+n} - \beta_n) \xi^{(k+1/2)}_{m,j+1/2} w_m, \tag{102}$$

$$\alpha_n = \left(\sum_{m \in M^{+}} \mu_m^{\alpha+n} w_m \right)^{-1}, \tag{103}$$

$$\beta = \frac{\alpha_0}{\alpha_n}. \tag{104}$$

Similarly, we derive the equations for variation of difference solutions from exact one (Eq. (86)) for the discretized α-WL methods. In this case, the same condition (92) must be satisfied to cancel out the O(1) terms. Therefore, the procedure described above is applied, and the functionals $P^{\pm}$ and $R^{\pm}$ are calculated in the following form:

$$P^{\pm}_{j+1/2} = \sum_{m \in M^{\pm}} (\alpha_0 |\mu_m|^{\alpha} - 1) \psi_{m,j+1/2} w_m. \tag{105}$$

$$R^{\pm}_{j+1/2} = \sum_{m \in M^{\pm}} (\alpha_1 |\mu_m|^{\alpha+1} - 1) \psi_{m,j+1/2} w_m. \tag{106}$$

As a result, the O(1) terms are satisfied. To make the resulting equations of the nonlinear and linear methods consistent with each other, we also replace γ (Eq. (34)) with β_1 [Eq. (104)] in the low-order equations of the α-WL methods (Eqs. (68)–(71)). Finally, the O(ε) equations of the discretized α-WL methods are identical to the O(ε) equations of the linearized α-WN methods (Eqs. (97)–(104)).

5.2. Fourier Analysis

We now apply a Fourier analysis to the discretized equations of the considered methods for a model infinite medium problem with constant cross sections and a uniform spatial mesh. We choose the following Fourier ansatz:

$$\xi^{(k+1/2)}_{m,j+1/2} = \alpha_m \omega^k e^{i\lambda\sigma_t x_{j+1/2}}, \tag{107}$$

$$\zeta^{\pm(k)}_{j+1/2} = \nu^{\pm} \omega^k e^{i\lambda\sigma_t x_{j+1/2}}, \tag{108}$$

$$\zeta^{\pm(k)}_{j} = t^{\pm} \omega^k e^{i\lambda\sigma_t x_{j}}, \tag{109}$$

$$\zeta^{(k)}_{j} = \omega^k e^{i\lambda\sigma_t x_{j}}. \tag{110}$$

Introducing the Fourier ansatz into the equations (91) and (97)–(104), we obtain a system of equations for α_m, $\nu^{\pm}$, $t^{\pm}$, and ω. Solving the resulting system for ω, we obtain

$$\omega = \frac{\omega_1}{\omega_2}, \tag{111}$$

where

$$\begin{aligned} \omega_1 = \sigma_t \sigma_s h^2 \big[& 2\beta_1 \delta_t \big(2\delta_a S_1 + \sigma_t^2 h^2 S_0 \big) \tan^2\chi + (\delta_t \delta_a + \upsilon \tan^2\chi) \\ & \times \big(\sigma_t^2 h^2 S_0 + 2(Z_1 - \beta_1 Z_0) \tan^2\chi \big) \big], \end{aligned} \tag{112}$$

$$\omega_2 = -\delta_t\delta_a\sigma_t\sigma_a h^2 - v(2\beta_1\delta_t + \sigma_t\sigma_a h^2)\tan^2\chi, \tag{113}$$

$$\chi = 0.5\lambda\alpha_t h, \tag{114}$$

$$\delta_a = 2\beta_1\theta + \sigma_a h, \tag{115}$$

$$\delta_t = 2\beta_1\theta + \sigma_t h, \tag{116}$$

$$v = 2\beta_1\theta\delta_a + \sigma_t\sigma_a h^2, \tag{117}$$

and for $n = 0$ and 1,

$$Z_n = \sum_{m\in M^+} g_{n,m}(2\mu_m + \sigma_t h(1 - 2T_m))w_m, \tag{118}$$

$$S_n = \sum_{m\in M^+} g_{n,m}w_m, \tag{119}$$

$$g_{n,m} = \frac{\alpha_0\mu_m^{\alpha+n} - \beta_n}{\sigma_t^2 h^2 + (2\mu_m + \sigma_t h(1 - 2T_m))^2\tan^2\chi}. \tag{120}$$

The spectral radius is defined by

$$\rho = \sup_{0\le\chi\le\frac{\pi}{2}} |\omega(\chi)|. \tag{121}$$

The spectral radius determines the stability of the considered methods.

5.3. Numerical Results

To obtain the spectral radii for various infinite medium problems, we evaluate $|\omega|$ using Eqs. (111)–(120). Table II contains theoretical estimates of the spectral radii ρ for the α-WN and α-WL methods versus $\sigma_t h$ and α, for $c = 1$, where $c = \sigma_s/\sigma_t$ is the scattering ratio. To calculate ω, we use the double S_5 Gauss–Legendre quadrature set. We notice that the spectral radius drastically decreases as $\sigma_t h$ increases. Except for optically thick cells, the methods with $\alpha = 0.128$ in the considered discretized form possess the minimum spectral radius. These theoretical results are consistent with the findings of the analysis of the α-weighted methods in differential form. The disagreement for optically thick cells is due to the utilized spatial discretization and angular quadrature set.

TABLE II
Theoretically Estimated Spectral Radii for the α-WN and α-WL Methods versus $\sigma_t h$ and α for c = 1

	α				
$\alpha_t h$	0	0.128	0.366	1	θ
0.01	0.33	0.19	0.22	0.31	3
0.1	0.33	0.19	0.22	0.31	3
1	0.21	0.11	0.15	0.26	3
2	0.10	0.049	0.084	0.20	3
3	0.045	0.022	0.051	0.13	3
5	0.0076	0.0061	0.017	0.039	1
10	6.5×10^{-5}	3.3×10^{-4}	4.6×10^{-4}	7.9×10^{-4}	1

TABLE III
Theoretically Estimated Spectral Radii for the α-WN and α-WL Methods versus $\sigma_t h$ and α for Problem 1 (c = 0.99)

	α				
$\alpha_t h$	0	0.128	0.366	1	θ
0.01	0.21	0.18	0.22	0.30	3
0.1	0.21	0.18	0.22	0.30	3
1	0.12	0.11	0.15	0.24	3
2	0.045	0.047	0.077	0.16	3
3	0.016	0.020	0.039	0.086	3
5	0.0016	0.0041	0.0096	0.021	1
10	6.7×10^{-6}	3.0×10^{-5}	7.8×10^{-5}	1.8×10^{-4}	1

We now present results that compare the theoretical values of spectral radii with numerically estimated values, obtained using the direct solution of transport problems by means of the considered methods.

Problem 1. We consider slab geometry problems: $\sigma_t = 1, \sigma_s = 0.99, Q = 1$, a reflecting boundary $x = 0$, and a vacuum boundary $x = L$. We use $L = 200h$, where h is the width of spatial cell. Thus, every problem consists of 200 equal spatial intervals. The angular mesh for the transport equation and quadrature weights for integration with respect to μ correspond to the double S_5 Gauss–Legendre quadrature set. Note that the use of such quadrature set is necessary, because all functionals (Eqs. (7)–(10) and (29)–(32)) consist of integrals over half-intervals. The convergence criterion has the form

$$\max_j \left| 1 - \frac{\phi_j^{(k)}}{\phi_j^{(k-1)}} \right| < \epsilon, \tag{122}$$

wherees $\epsilon = 10^{-7}$, j spans the spatial mesh, and k is the iteration index.

Tables III–V contain the theoretically and numerically estimated spectral radii for the α-WN and α-WL methods. The numerically estimated spectral radii were determined by

TABLE IV
Numerically Estimated Spectral Radii for the α-WN Methods versus $\sigma_t h$ and α for Problem 1 (c = 0.99)

	α				
0.01	0.11	0.10	0.14	0.24	3
0.1	0.19	0.13	0.17	0.23	3
1	0.090	0.085	0.12	0.20	3
2	0.035	0.036	0.056	0.13	3
3	0.012	0.014	0.026	0.069	3
5	8.0×10^{-4}	0.0022	0.0054	0.014	1
10	3.4×10^{-6}	9.1×10^{-6}	2.5×10^{-5}	5.4×10^{-5}	1

TABLE V
Numerically Estimated Spectral Radii for the α-WL Methods versus $\sigma_t h$ and α for Problem 1 (c = 0.99)

	α				
$\sigma_t h$	0	0.128	0.366	1	θ
0.01	0.15	0.16	0.21	0.28	3
0.1	0.18	0.16	0.20	0.27	3
1	0.10	0.092	0.12	0.22	3
2	0.037	0.038	0.063	0.14	3
3	0.012	0.015	0.031	0.073	3
5	0.0011	0.0028	0.0065	0.016	1
10	2.9×10^{-6}	1.5×10^{-5}	4.1×10^{-5}	9.6×10^{-5}	1

using the formula

$$\rho = \frac{\left\|\phi^{(k)} - \phi^{(k-1)}\right\|_{L_2}}{\left\|\phi^{(k-1)} - \phi^{(k-2)}\right\|_{L_2}} \tag{123}$$

for the last iteration in each problem. The results show that the Fourier analysis approximately predicts the value of the spectral radii for the α-WN and α-WL methods. As may be seen from the numerical results, the considered nonlinear and linear methods have similar convergence properties. Note that for $c = 1$, the methods with $\alpha = 0$ possess the maximum spectral radii for optically thin cells among the methods under examination (Table I). However, for $c = 0.99$, the methods with $\alpha = 0$ are second only to the methods with $\alpha = 0.128$ for optically thin cells, and moreover they have the minimum spectral radii for the other optical thicknesses. Such changes in the spectral radii of the considered methods are predicted theoretically (Table III) and are confirmed by numerical calculations (Tables IV and V).

These theoretical and numerical estimations of the convergence properties were obtained for the class of problems in which the considered iterates are close to the solution. Next, we compare the α-weighted methods in general cases considering directly the iteration numbers in problems with scattering ratios close to unity.

Problem 2. We consider a slab $0 \leq x \leq 20$ having $\sigma_t = 1$, $\sigma_s = 0.97$, $Q = 0$. The left boundary has an isotropic incident flux with magnitude unity, and the right boundary is reflecting [24]. A spatial mesh consisting of J equal cells with $h = 20/J$ is used. The double S_4 Gauss–Legendre quadrature set is used. The relative pointwise convergence criterion (122) with $\epsilon = 10^{-12}$ is imposed.

The number of iterations are listed in Table VI. These results demonstrate that the methods with $\alpha = 0.128$ converge faster than other ones, and that the considered linear and nonlinear methods have almost equal convergence rates in problems with isotropic scattering. Comparing the DSA-like methods ($\alpha = 0.366$) with the DSA and QD methods, we note that these methods have approximately the same numbers of iterations for optically thin cells. However, for optically thick cells the α-WN and α-WL methods with $\alpha = 0.366$ have advantages over the QD and DSA methods in convergence rates. The iteration numbers of the DSA method were taken from [24].

In the next test problem, the issues related to accuracy of the α-weighted methods and pure SC method are considered.

TABLE VI
Number of Iterations of the α-WN and α-WL Methods versus $\sigma_t h$ and α for Problem 2

	α-WN				α-WL					
	α				α					
$\sigma_t h$	0	0.128	0.366	1	0	0.128	0.366	1	DSA	QD
5	5	5	7	8	5	5	6	7	14	10
2	10	9	11	16	9	9	11	14	16	14
1	13	12	13	19	12	11	13	18	16	14
0.5	15	13	15	20	14	13	15	19	16	15
0.25	16	14	16	20	15	14	15	20	16	15
0.125	16	14	16	20	16	14	16	20	16	15

Problem 3. We consider a slab $0 \leq x \leq 20$ with $\sigma_t = 1$ $\sigma_s = 1$, $Q = 10^{-2}$, a reflecting boundary $x = 0$, and a vacuum boundary $x = L$. We use uniform spatial meshes with 10 and 20 intervals, i.e., the optical thickness of intervals of these meshes are $\sigma_t h = 2$ and $\sigma_t h = 1$, respectively. The double S_5 Gauss–Legendre quadrature set is utilized. The relative pointwise convergence criterion (122) with $\epsilon = 10^{-7}$ is imposed.

Figures 1 and 2 demonstrate the numerical solution (ϕ) of the SC method (obtained with source iterations) and α-WN methods with different values of the parameter α. Let us

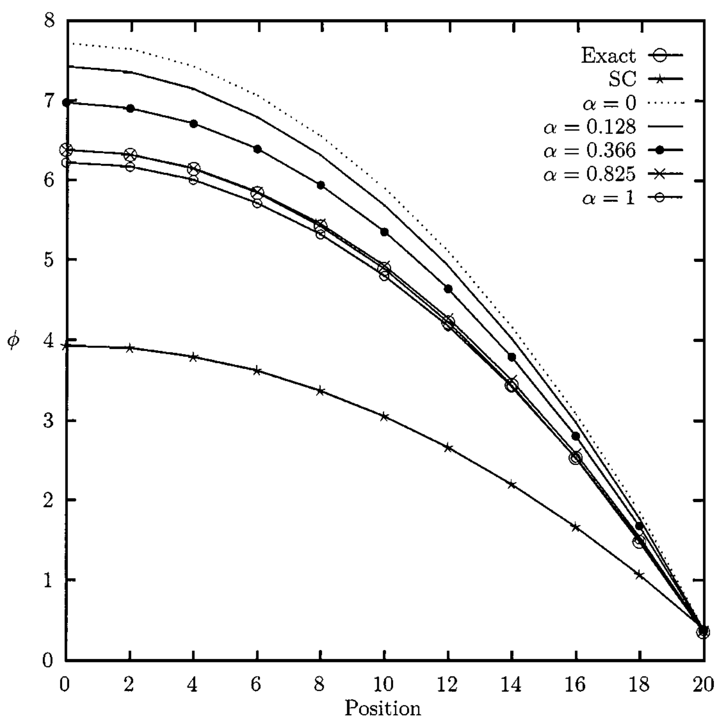

FIG. 1. Problem 3. The scalar flux ϕ obtained by the SC method and α-WN methods with various α in case $\sigma_t h = 2$.

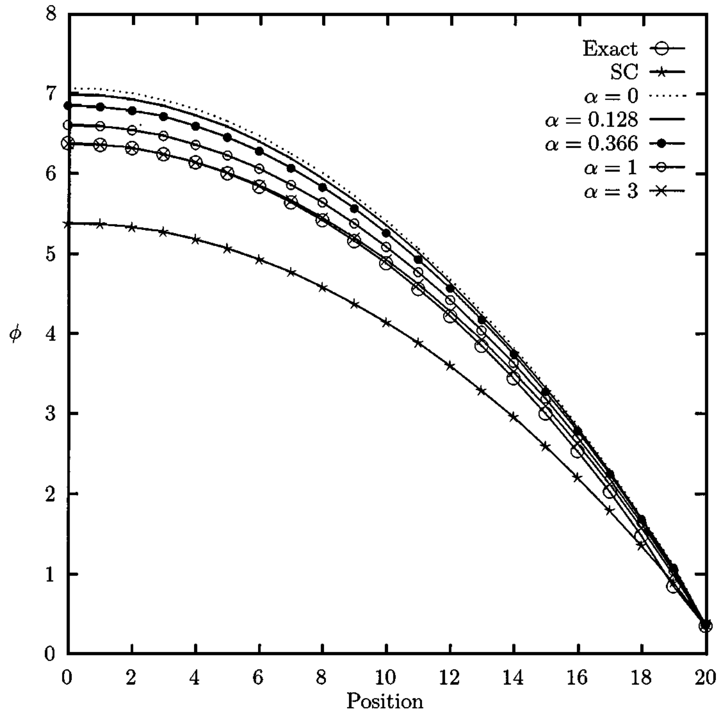

FIG. 2. Problem 3. The scalar flux ϕ obtained by the SC method and α-WN methods with various α in case $\sigma_t h = 1$.

recall that there are two scalar flux solutions that come out of the α-weighted methods. The presented scalar flux of the α-WN methods is the one that is the solution of the low-order problem. The exact solution was obtained by the QD method on fine spatial and angular meshes. These results show that the solution of the α-weighted methods is more accurate than the solution of the SC method itself, in spite of the fact that in the framework of the α-weighted methods the SC method is used to calculate the angular flux and generate the functionals. The scalar flux solutions of the α-WN methods from the high-order problem calculated by the SC method are very close to those from the low-order problem. We discuss only the results of the nonlinear methods because the behavior of the numerical solution of the α-WL methods is similar.

We notice that the accuracy of the α-weighted methods depends on the value of α. Table VII contains the relative errors of numerical solutions at $x = 0$ obtained by the α-WN

TABLE VII
The Relative Errors [%] of Numerical Solutions of the α-WN Methods at $x = 0$ versus $\sigma_t h$ and α for Problem 3

	α							
$\sigma_t h$	0	0.128	0.366	1	2	4	10	30
1	−10.7	−9.5	−7.4	−3.5	−1.0	0.5	1.0	1.1
2	−20.9	−16.4	−9.3	2.5	10.5	16.1	19.3	20.2

methods with various values of α. The analysis of errors of numerical results enabled us to determine the "optimal" value of α with which the α-WN method generates an accurate numerical solution of the given test problem on a particular mesh. The α-WN method with $\alpha = 0.825$ gives the most accurate solution on the mesh with $\sigma_t h = 2$. The relative error at $x = 0$ in this case is very small and equal to -0.04%. The α-WN method with $\alpha = 3$ produces the most accurate solution on the mesh with $\sigma_t h = 1$. The relative error at $x = 0$ is equal to 0.04%. Note that among a variety of the α-weighted methods there are three methods with some important specific features of the low-order equations. The low-order equations of the method with $\alpha = 0$ (the FF method) have a form of the particle balance equation. The low-order equations in case of $\alpha = 1$ (the SF method) result in the Fick's law in optically thick regions. In case of $\alpha = \alpha_{DSA}$, the low-order equations reduce to the diffusion equation in the optically thick regions. The presented numerical solutions indicate that none of these methods is the most accurate on intermediate meshes.

Thus, the results of Problem 3 show that the α-weighted methods are clearly more accurate than the pure SC method for all values of α. This demonstrates the improvement of accuracy that is possible in case of independent discretization. The optimization of accuracy by means of choosing special weight function is another specific option of the proposed α-weighted methods. It is interesting to obtain theoretical results that will describe the relatinship between the value of α and accuracy of these methods.

6. DISCUSSION

We have proposed a new family of nonlinear and linear methods for solving particle transport problems. To discretize these methods, one can use independent schemes for the transport and low-order equations. The proposed nonlinear α-weighted method reduces to the FF or SF method for special values of α. The α-WL methods have the same mathematical form as the linearized α-WN methods. The proposed methods have been presented in slab geometry.

We studied theoretically the convergence properties of these methods in both differential and discretized form for transport problems with isotropic scattering. We showed that the considered nonlinear and linear methods have identical convergence behavior in the vicinity of the solution. The Fourier analysis of the α-WL and linearized α-WN methods in differential form revealed that the methods with $\alpha = 0.128$ possess the best convergence rates, and the methods with $\alpha \approx 0.366$ have convergence properties similar to the DSA and linearized QD methods. The theoretical analysis of the discretized methods enables us to predict approximately their spectral radii. The considered methods were approximated by means of independent differencing schemes. We used the linear discontinuous method for the low-order equations and the step characteristic method for the transport equation. Numerical testing confirmed that the nonlinear and linear methods have almost equal iteration numbers in problems with isotropic scattering. We also showed the advantage of the methods with $\alpha = 0.128$ in convergence rates for optically thin cells.

The low-order equations of the α-weighted methods are similar in structure to the transport equation and, hence, transport differencing schemes can be used as a basis for their approximation. The α-weighted methods in multidimensional geometries can be good for solving the transport equation in parallel, because parallel algorithms for the transport equation can be applied to the low-order equations of these methods. Note that these features of the α-weighted methods are close philosophically in some aspects to the transport synthetic acceleration (TSA) method [26]. There is an interesting option of developing new iterative methods based on some combination of ideas of the α-weighted and TSA methods.

The derivation of the proposed methods in multidimensional geometries is similar to the way the flux methods are formulated [10, 27]. It is based on dividing the unit sphere on nonoverlaping subsets and integrating the transport equation over them. The α-weighted methods as well as the flux methods can be applied for the problems with anisotropic scattering. In such a case, the right-hand side of the transport equation should be also transformed by means of moments of the transport solution and set of stable functionals, according to methodology developed in the QD method [1, 12, 15].

In one-dimensional geometry the discretized low-order equations of the α-weighted methods can be easily solved. In case of multidimensional geometries, efficient iteration methods must be used. Krylov subspace methods are good candidates for solving these equations. This is an issue for future research.

The present paper addresses the stability analysis of the proposed parametric family of iteration methods for solving the transport equation. In further research we plan to consider various aspects of approximation of high- and low-order equations and approaches for efficient solving the low-order problem of the α-WN methods. It is interesting to consider a parametrized family of nonlinear and linear methods with a different set of weight functions, for example, $w(\mu) = 1 + \zeta|\mu|^\alpha$, where α and ζ are constants. Preliminary investigations have shown that the method with $\alpha = 1$ and $\zeta = \sqrt{3}$ has low-order equations that result in the diffusion equation in case of a nearly isotropic angular flux, i.e., in the diffusion limit. This particular method possesses some other good features as well, which make approximation of low-order equations easier. Another set of weight functions that deserves to be studied is general polynomials of μ.

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